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Nearly Comonotone Approximation*

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We discuss the degree of approximation by polynomials of a function f that is piecewise monotone in $[-1, 1]$. We would like to approximate f by polynomials which are comonotone with it. We show that by relaxing the requirement for comonotonicity in small neighborhoods of the points where changes in monotonicity occur and near the endpoints, we can achieve a higher degree of approximation. We show here that in that case the polynomials can achieve the rate of ω_3. On the other hand, we show in another paper, that no relaxing of the monotonicity requirements on sets of measures approaching 0 allows ω_4 estimates.

1. INTRODUCTION

Let $I := [-1, 1]$, and for $s \geqslant 1$ let $Y := \{y_i\}_{i=0}^{s}$, $-1 = y_s < \cdots < y_1 < y_0 = 1$. Finally let $\Delta^{(1)}(Y)$ be the set of continuous functions f on I, such that f is nondecreasing on $[y_i, y_{i-1}]$, when i is odd and it is nonincreasing on $[y_i, y_{i-1}]$, when i is even, and set

$$\Pi(x) := \prod_{i=1}^{s-1} (x - y_i).$$

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A polynomial P_n is said to be comonotone with $f \in \Delta^{(1)}(Y)$ on the set $E \subset I$, if $P_n'(x)\,\Pi(x) \geqslant 0$, $x \in E$. Note that if $f \in C^1(-1, 1)$, then $f'(x)\,\Pi(x) \geqslant 0$, $x \in (-1, 1)$ if and only if $f \in \Delta^{(1)}(Y)$.

A. S. Shvedov [10] proved that for each Y there exists a constant $c(Y)$, such that for every $f \in \Delta^{(1)}(Y)$ and all $n \geqslant 1$ an algebraic polynomial P_n, of degree $\leqslant n$, which is comonotone with f on I, exists satisfying

$$\|f - P_n\|_{C(I)} \leqslant c(Y)\,\omega_2(f, 1/n), \tag{1.1}$$

where $\omega_k(f; \cdot)$ denotes the modulus of smoothness of order k, of f. (Earlier DeVore [2] proved (1.1) for the case $s = 1$, which is the case where f is monotone and of course the dependence of $c(Y)$ on Y is meaningless, i.e., c is an absolute constant.) More recently, R. A. DeVore and X. M. Yu [4] and G. A. Dzyubenko [6] have shown that one can get also pointwise estimates, namely,

$$|f(x) - P_n(x)| \leqslant c(Y)\,\omega_2(f, \rho_n(x)), \tag{1.1$'$}$$

where $\rho_n(x) := \sqrt{1 - x^2}/n + 1/n^2$.

On the other hand it is known (see [10]) that in (1.1) and (1.1′), one cannot replace ω_2 by ω_k with any $k \geqslant 3$.

It is quite natural to ask whether one can strengthen (1.1) in the sense of being able to replace ω_2 by moduli of smoothness of higher order, if one is willing to allow P_n not to be comonotone with f on a rather "small" subset of I. This indeed turns out to be possible for ω_3, as we show in Theorem 1. However, even this improvement comes to a halt, it cannot be extended to ω_4, and thus not to ω_k for any $k > 3$. We devote a separate paper [3] to proving this assertion when f is monotone. Here we will only state the result in the general case (see Theorem 4 below); the proof is a modification of [3].

We begin with some notation needed for the statement of Theorem 1. Let $x_{-1} := 1$, $x_{n+1} := -1$ and for each $j = 0, ..., n$, set $x_j := x_{j,n} := \cos(j\pi/n)$, $I_j := I_{j,n} := [x_j, x_{j-1}]$, and $h_j := h_{j,n} := |I_j| := x_{j-1} - x_j$. For later reference we need the following well known relations (see, e.g., [7])

$$\rho_n(x) < h_j < 5\rho_n(x), \qquad x \in I_j, \tag{1.2}$$

$$h_{j\pm1} < 3h_j, \qquad j = 1, ..., n, \tag{1.3}$$

$$\rho_n^2(y) < 4\rho_n(x)(|x - y| + \rho_n(x)), \qquad x, y \in I, \tag{1.4}$$

which in turn implies

$$2(|x - y| + \rho_n(x)) > |x - y| + \rho_n(y), \qquad x, y \in I, \tag{1.5}$$

and

$$c\left(\frac{h_j}{|x-x_j|+h_j}\right)^2 \leqslant \frac{\rho_n(x)}{|x-x_j|+\rho_n(x)}, \leqslant C\left(\frac{h_j}{|x-x_j|+h_j}\right)^{1/2}. \tag{1.6}$$

Given Y, let

$$O_i := O_{i,n}(Y) := (x_{j+1}, x_{j-2}), \qquad \text{if} \quad y_i \in [x_j, x_{j-1}),$$

$$O(n; Y) := \bigcup_{i=1}^{s-1} O_i, \quad n \geqslant 1, \qquad O(0, Y) := [-1, 1],$$

and

$$O^*(n, Y) := O(n, Y) \cup I_1 \cup I_n, \quad n \geqslant 1, \qquad O^*(0, Y) := [-1, 1].$$

We first prove

THEOREM 1. *There are constants $c = c(s)$ and $C(s)$ for which, if $f \in \Delta^{(1)}(Y)$, then for every $n > 1$, a polynomial P_n of degree not exceeding n, which is comonotone with f on $I \backslash O^*([n/c], Y)$ exists, such that*

$$|f(x) - P_n(x)| \leqslant C(s)\, \omega_3(f, \rho_n(x)). \tag{1.7}$$

We are able to obtain estimates involving moduli of higher orders for classes of differentiable functions, namely,

THEOREM 2. *Let $k \geqslant 2$ be fixed. Then there are constants $c = c(s, k)$ and $C(s, k)$ for which, if $f \in \Delta^{(1)}(Y) \cap C^1[-1, 1]$, then for each $n \geqslant k-1$, a polynomial P_n of degree not exceeding n, which is comonotone with f on $I \backslash O([n/c], Y)$, exists such that*

$$|f(x) - P_n(x)| \leqslant C(s, k)\, \rho_n(x)\, \omega_{k-1}(f', \rho_n(x)), \qquad x \in I. \tag{1.8}$$

It is interesting to note that the differentiability of f, without giving up some small neighborhoods of the points Y, in general does not allow statements like (1.8). Indeed among others, it is shown in [8] that there is an $f \in \Delta^{(1)}(Y) \cap C^1[-1, 1]$, with $s > 1$ number of changes of monotonicity, which is thrice differentiable in $(-1, 1)$ and such that $(1-x^2)^{3/2} f^{(3)}(x)$ is bounded there, and yet the least distance (in the sup-norm) between f and polynomials which are truly comonotone with it, can be made as large as one wishes. (There are other interesting phenomena for truly comonotone approximations; the interested reader is referred to [8].)

Observe that $O(1, Y) = O^*(1, Y) = [-1, 1]$, thus it is clear that for $n < 2c$, we place no monotonicity constraint on the approximating polynomials. Therefore Theorems 1 and 2 follow from the well-known unconstrained Timan–Dzjadyk–Freud–Brudnyi estimates and we only have to prove them for larger n.

We shall make no attempt to estimate how small the constants c in the above theorems can be. Obviously, the smaller they are the smaller the neighborhoods $O^*([n/c], Y)$ and $O([n/c], Y)$ are, thus the stronger the results are. However, we feel it is important to point out that c cannot be too small or the above theorems become false for $s > 2$. To this end we prove the following result in Section 5.

THEOREM 3. *For each $A \geqslant 1$ and $n \geqslant 60A$, there exists a collection $Y_n := \{y_i\}_{i=0}^{3}$, and a function $f = f_n \in \Delta^{(1)}(Y_n)$, such that any polynomial P_n of degree not exceeding n which satisfies*

$$P_n'(x)\, f'(x) \geqslant 0, \qquad x \notin O^*(8\pi n, Y_n),$$

necessarily satisfies also

$$\left\| \frac{f - P_n}{\omega_1(f, \rho_n(\cdot))} \right\| > A. \tag{1.9}$$

Note that the collection Y_n depends on A and that if we stated Theorems 1 and 2 with constants that depend on Y, then obviously we would not have the analogue of Theorem 3. Also, evidently when A increases, n is taken bigger and bigger. Indeed, for small (fixed) n, it is possible to take c in Theorems 1 and 2 as small as we wish if we are willing to pay by enlarging C. Furthermore, if $s = 1$ or 2, then it is possible to take c arbitrarily small (at the expense of increasing C). Finally one should note that for any s, the neighborhoods of the endpoints in Theorem 1 can be taken to be of length of arbitrarily small (ε) proportion of $1/n^2$ while allowing $C := C(s, \varepsilon)$.

To conclude this section, we state without proof (the proof will be given elsewhere) the following result which asserts that Theorem 1 cannot be valid for higher moduli of smoothness (see [3]). To this end, given $\varepsilon > 0$ and a function $f \in \Delta^{(1)}(Y)$, we denote

$$E_n^{(1)}(f; \varepsilon) := \inf_{P_n} \| f - P_n \|_{C(I)},$$

where the infimum is taken over all polynomials P_n of degree not exceeding n satisfying

$$\text{meas}(\{x;\ P_n'(x)\ \Pi(x) \geqslant 0\} \cap I) \geqslant 2-\varepsilon.$$

THEOREM 4. *For each sequence* $\bar{\varepsilon} = \{\varepsilon_n\}_{n=1}^{\infty}$, *of nonnegative numbers tending to* 0, *there exists a function* $f := f_{\bar{\varepsilon}} \in \Delta^{(1)}(Y)$, *such that*

$$\limsup_{n\to\infty} \frac{E_n^{(1)}(f;\varepsilon_n)}{\omega_4(f, 1/n)} = \infty. \tag{1.10}$$

Throughout this paper we take $k \geqslant 2$. In the sequel we will have constants which depend on s and k. If they are independent of any other term, then we will not explicitly write this dependence. However, we will use the notation c and C to denote such constants which are of no significance to us and may differ on different occurrences, even in the same line; and we will have constants with indices $c_1, c_2, ...$ and $C_1, C_2, ...$ when we have a reason to keep trace of them in the computations that we have to carry in the proofs.

2. AUXILIARY LEMMAS

Since we deal with functions f, which are piecewise monotone, then f' exists a.e. in I. We will use the max-norm of f as well as the norm of f' in L_∞ (when applicable). Thus, for any interval $J \subset I$, let us denote

$$\|f\|_J := \|f\|_{L_\infty(J)},$$

which is obviously compatible with the max-norm whenever f is continuous.

Throughout this section, $n \geqslant 1$ is going to be fixed so that we would not have to carry n as an index for the intervals, etc.

First we prove

LEMMA 1. *Let* $H_0 := 0 < H_1 < H_2 < H_3$, $\eta_j := H_j - H_{j-1}$, $j = 1, 2, 3$, *be such that* $1/3 < \eta_2/\eta_j < 3$, $j = 1, 3$; *and let* $f \in C[0, H_3]$ *be nondecreasing in* $[0, H_3]$. *Then there is a quadratic polynomial* P_2, *interpolating* f *at* H_1 *and* H_2, *such that*

$$\|f - P_2\|_{[H_1, H_2]} \leqslant c\omega_3(f, \eta_2; [0, H_3]),$$

and

$$P_2'(x) \geqslant 0, \qquad x \in [H_1, H_2].$$

Proof. Let

$$L_0(x) := L(x; f; 0, H_1, H_2) \qquad \text{and} \qquad L_2(x) := L(x; f; H_1, H_2, H_3)$$

be the Lagrange polynomials of degree 2, interpolating f at the points $0, H_1, H_2$, and H_1, H_2, H_3, respectively, and let $L_1(x) := L(x; f; H_1, H_2)$ be the linear function which interpolates f at H_1 and H_2.

If $L_0'(x) \geqslant 0$, or $L_2'(x) \geqslant 0$, for $x \in [H_1, H_2]$, then the assertion follows from Whitney's inequality. Otherwise, we have

$$L_0''(x) \leqslant 0 \qquad \text{and} \qquad L_2''(x) \geqslant 0, \qquad x \in [H_1, H_2],$$

whence

$$L_2(x) \leqslant L_1(x) \leqslant L_0(x), \qquad x \in [H_1, H_2].$$

Applying Whithey's inequality we get, for $x \in [H_1, H_2]$,

$$\begin{aligned} L_1(x) - f(x) \leqslant L_0(x) - f(x) &\leqslant \|L_0 - f\|_{[H_1, H_2]} \\ &\leqslant \|L_0 - f\|_{[0, H_3]} \leqslant c\omega_3(f, \eta_2; [0, H_3]), \end{aligned}$$

and

$$\begin{aligned} f(x) - L_1(x) \leqslant f(x) - L_2(x) &\leqslant \|f - L_2\|_{[H_2, H_2]} \\ &\leqslant \|f - L_2\|_{[0, H_3]} \leqslant c\omega_3(f, \eta_2; [0, H_3]). \end{aligned}$$

Hence

$$\|L_1 - f\|_{[H_1, H_2]} \leqslant c\omega_3(f, \eta_2). \quad \blacksquare$$

Next we have

LEMMA 2. *If $f \in C^1[0, h]$ and $f'(x) \geqslant 0$ for $x \in [0, h]$, then there is a polynomial P_{k-1} of degree $\leqslant k-1$ such that*

$$\|f - P_{k-1}\|_{[0, h]} \leqslant ch\omega_{k-1}(f', h; [0, h]), \tag{2.1}$$

$$f(0) = P_{k-1}(0), \qquad f(h) = P_{k-1}(h) \tag{2.2}$$

and

$$P_{k-1}'(x) \geqslant 0, \qquad x \in [0, h]. \tag{2.3}$$

Proof. If f is a polynomial of degree $<k$, then obviously there is nothing to prove. Otherwise let P^*_{k-2} be the polynomial of the best uniform approximation of f' in $[0, h]$, and denote

$$E_{k-2} := \|f' - P^*_{k-2}\|_{[0,h]} > 0.$$

Set

$$P_{k-1}(x) := f(0) + \frac{f(h)-f(0)}{\int_0^h (P^*_{k-2}(u)+E_{k-2})\,du} \int_0^x (P^*_{k-2}(u)+E_{k-2})\,du,$$

where it is readily seen that the denominator is not zero and that (2.3) follows by

$$P^*_{k-2}(u)+E_{k-2} \geqslant f'(u) \geqslant 0, \qquad u \in [0, h].$$

Also, it is evident that (2.2) holds thus we only have to prove (2.1). To this end, for $x \in [0, h]$ we have

$$\begin{aligned} f(x) - P_{k-1}(x) &= \int_0^x (f'(u) - P^*_{k-2}(u) - E_{k-2})\,du - \int_0^h (f'(u) - P^*_{k-2}(u) - E_{k-2})\,du \\ &\quad \times \frac{\int_0^x (P^*_{k-2}(u)+E_{k-2})\,du}{\int_0^h (P^*_{k-2}(u)+E_{k-2})\,du}. \end{aligned}$$

Hence

$$\|f - P_{k-1}\|_{[0,h]} \leqslant 2hE_{k-2}.$$

Now (2.1) follows by Whitney's inequality. ∎

Denote by Σ_k the collection of continuous piecewise polynomials of degree $\leqslant k$ with the knots at the x_j's. Thus, $S \in \Sigma_k$ is differentiable in I except perhaps at the x_j's. We denote this derivative by S'.

Let $\varphi \in \Phi^k$, i.e., $\varphi(0+) = 0$ and $\varphi(t)$ is nondecreasing while $t^{-k}\varphi(t)$ is nonincreasing on $(0, \infty)$. We will use the ordinary notation $f \in H_k^\varphi$ and $f \in W^1H_k^\varphi$, respectively, for f with $\omega_k(f, \cdot) \leqslant \varphi$ and for differentiable f with $\omega_k(f', \cdot) \leqslant \varphi$.

Then, Lemmas 1 and 2 readily imply the following Lemmas 3 and 4, respectively.

LEMMA 3. *Let $\varphi \in \Phi^3$. If $f \in H_3^\varphi$ and $f \in \Delta^{(1)}(Y)$, then there is an $S \in \Sigma_2$ such that*

$$\|f - S\|_{I_j} \leqslant c\varphi(h_j), \qquad j = 1, ..., n,$$

and

$$S'(x)\,\Pi(x) \geqslant 0 \qquad \text{in } I\backslash O^*.$$

LEMMA 4. *Let $\omega \in \Phi^{k-1}$ and $\varphi(t) := t\omega(t)$. If $f \in W^1 H^{\omega}_{k-1} \cap \Delta^1(Y)$, then there is an $S \in \Sigma_{k-1}$ such that*

$$\|f - S\|_{I_j} \leqslant c\varphi(h_j), \qquad j = 1, ..., n,$$

and

$$S'(x)\,\Pi(x) \geqslant 0 \qquad \text{in } I\backslash O.$$

The following lemma is proved very much like [7, Lemma 5.4] (see also a simpler variant [8, Theorem 1]).

LEMMA 5. *Let $\varphi \in \Phi^k$ and suppose that f is locally absolutely continuous, that*

$$\|f'\|_{I_j} \leqslant \frac{1}{h_j}\,\varphi(h_j), \qquad j = 1, ..., n$$

and that

$$f'(x)\,\Pi(x) \geqslant 0, \qquad \text{a.e.} \quad x \notin O \quad \text{or} \quad x \notin O^*.$$

Then there is a polynomial V_n such that

$$\|f - V_n\|_{I_j} \leqslant c\varphi(h_j), \qquad j = 1, ..., n$$

and

$$V_n'(x)\,\Pi(x) \geqslant 0, \qquad x \notin O \quad \text{or} \quad x \notin O^*,$$

respectively.

Now let $I_{i,j}$ be the smallest interval containing I_i and I_j and denote $h_{i,j} := |I_{i,j}|$. For $S \in \Sigma_{k-1}$, put

$$a_{i,j} = a_{i,j}(S, \varphi) := \frac{\|p_i - p_j\|_{I_i}}{\varphi(h_j)} \left(\frac{h_j}{h_{i,j}}\right)^k, \qquad i, j = 1, ..., n, \tag{2.4}$$

where p_i is the polynomial defined by $p_i|_{I_i} := S|_{I_i}$. Finally for any $E \subset I$ let

$$a_k(S, \varphi; E) := \max a_{i,j}(S, \varphi),$$

where the maximum is taken over all i, j such that $\mathring{I}_i \cap E \neq \varnothing$ and $\mathring{I}_j \cap E \neq \varnothing$, where $\mathring{J}$ denotes the interior of J; and

$$a_k := a_k(S, \varphi; I).$$

We have

LEMMA 6. *There is a constant c, depending only on k, such that for any $f \in H_k^\varphi$ and $S \in \Sigma_{k-1}$, if*

$$\|f - S\|_{I_j} \leqslant \varphi(h_j), \qquad j = 1, ..., n, \tag{2.5}$$

then

$$a_k \leqslant c. \tag{2.6}$$

Proof. We divide I_j into k subintervals of equal lengths by setting $x_{j,0} := x_j < x_{j,1} < \cdots < x_{j,k-1} < x_{j-1}$ and we let L_k be the Lagrange polynomial of degree $k-1$ interpolating f at $x_{j,l}$, $l = 0, ..., k-1$. Then by Whitney's theorem

$$\|f - L_k\|_{I_j} \leqslant c\omega_k(f, h_j) \leqslant c\varphi(h_j). \tag{2.7}$$

Hence by (2.5)

$$\|p_j - L_k\|_{I_j} \leqslant c\varphi(h_j),$$

which implies

$$\|p_j - L_k\|_{I_i} \leqslant c \left(\frac{h_{i,j}}{h_j}\right)^k \varphi(h_j). \tag{2.8}$$

At the same time, (2.7) implies (see [9, p. 51, (4.15)])

$$\|f - L_k\|_{I_i} \leqslant c \left(\frac{h_{i,j}}{h_j}\right)^k \varphi(h_j). \tag{2.9}$$

Combining (2.5) with (2.8) and (2.9) we obtain

$$\begin{aligned}\|p_i - p_j\|_{I_i} &\leqslant c \left(\frac{h_{i,j}}{h_j}\right)^k \varphi(h_j) + \varphi(h_i)\\ &\leqslant c \left(\frac{h_{i,j}}{h_j}\right)^k \varphi(h_j),\end{aligned}$$

where if $h_j > h_i$ we used the inequality $\varphi(h_i) \leqslant \varphi(h_j)$, and if $h_j \leqslant h_i$, then due to $\varphi \in \Phi^k$ we have

$$\varphi(h_i) \leqslant \left(\frac{h_{i,j}}{h_i}\right)^k \varphi(h_i) \leqslant \left(\frac{h_{i,j}}{h_j}\right)^k \varphi(h_j). \quad \blacksquare$$

3. THE MAIN LEMMAS

We begin with a well-known partition of unity by polynomials which goes back to G. Freud and Yu. A. Brudnyi (see, e.g., Dzjadyk [5, p. 273–277]). For each fixed integer r, a collection $\{\tau_{j,n}\}_{j=1}^{n}$, of polynomials of degree $\leqslant n$, exists such that

$$\sum_{j=1}^{n} \tau_{j,n}(x) \equiv 1, \tag{3.1}$$

and for $q = 0, 1, \ldots$ we have

$$|\tau_{j,n}^{(q)}(x)| \leqslant C \frac{h_j}{\rho_n^{q+1}(x)} \left(\frac{\rho_n(x)}{|x - x_j| + \rho_n(x)}\right)^{r+1}, \qquad x \in I, \tag{3.2}$$

where C depends on q and r. (Inequality (3.2) for $q = 0$ follows from [5, p. 277, (13)], by (1.4) and (1.6); and for higher q by induction. Actually, we only need $q = 0, 1$.)

First we prove

Lemma 7. *Let $r \geqslant 3k$, $\varphi \in \Phi^k$, and $S \in \Sigma_{k-1}$. For $n_1 \geqslant N$, with n_1 divisible by n, the polynomial*

$$D_{n_1}(x) := \sum_{i=1}^{n} p_i(x) \sum_{\nu : I_{\nu, n_1} \subseteq I_i} \tau_{\nu, n_1}(x), \tag{3.3}$$

satisfies

$$|S(x) - D_{n_1}(x)| \leqslant C_0 a_k \varphi(\rho_n(x)), \qquad x \in I, \tag{3.4}$$

and

$$|S'(x) - D'_{n_1}(x)| \leqslant C_0 a_k \frac{\varphi(\rho_n(x))}{\rho_n(x)}, \qquad x \in I. \tag{3.5}$$

Moreover, for each $0<\delta<1$,

$$|S'(x)-D'_{n_1}(x)| \leqslant C_1 a_k(S, \varphi; (x-\delta, x+\delta)\cap I)\,\frac{\varphi(\rho_n(x))}{\rho_n(x)}$$

$$+c_2 a_k \left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x)+\delta}\right)^{r+1-3k} \frac{\varphi(\rho_n(x))}{\rho_n(x)}, \qquad x\in I, \qquad (3.6)$$

where $C_l = C_l(k, r)$.

Proof. Recall that throughout the paper we assume $k\geqslant 2$. (Since for $k=1$, S is a constant, Lemma 7 is valid also for $k=1$.) We will only prove (3.6), the proof of (3.4) being similar, and evidently, (3.5) being an immediate consequence of (3.6).

We fix $1\leqslant j\leqslant n$, and $x\in I_j$ and to save in writing we set $\rho := \rho_n(x)$, and $\rho_1 := \rho_{n_1}(x)$. Since $p_j - p_i$ is a polynomial of degree not exceeding $k-1$, then

$$\|p_j - p_i\|_{I_j} \leqslant c\left(\frac{h_{i,j}}{h_i}\right)^{k-1} \|p_j - p_i\|_{I_i}.$$

Hence by (1.6) and (2.4),

$$\|p_j - p_i\|_{I_j} \leqslant c\left(\frac{h_{i,j}}{h_i}\right)^{k-1} \left(\frac{h_{i,j}}{h_j}\right)^{k} \varphi(h_j)\, a_{i,j}$$

$$\leqslant c a_{i,j}\varphi(h_j)\left(\frac{h_{i,j}}{h_j}\right)^{3k-2} =: c\Omega_{i,j}, \qquad (3.7)$$

which in turn implies

$$\|p'_j - p'_i\|_{I_j} \leqslant \frac{c}{h_j}\,\Omega_{i,j}. \qquad (3.8)$$

(Note that for $u\in I_i$, (1.2) and (1.3) imply that $h_{i,j}\sim |u-x|+\rho$, that is, there are constants $0<c<C$ independent of i, j, and n, for which $ch_{i,j}<|u-x|+\rho<Ch_{i,j}$.) Now, if we write $|x-x_{i*}| := \min\{|x-x_i|, |x-x_{i-1}|\}$, then it follows by (3.7) and (3.8) that

$$|p_j(x)-p_i(x)| \leqslant c\,\frac{|x-x_{i*}|}{h_j}\,\Omega_{i,j}. \qquad (3.9)$$

Indeed if $i=j$, there is nothing to prove; if $i\neq j\pm 1$, then (3.9) is an immediate consequence of (3.7) and the inequality (see (1.3))

$$|x-x_{i*}| > h_j/3,$$

and if $i = j+1$ (a similar proof applies to $i = j-1$), then

$$|p_j(x) - p_i(x)| = \left| \int_{x_j}^{x} (p_j'(u) - p_i'(u))\, du \right|$$
$$\leqslant |x - x_j| \frac{c}{h_j} \Omega_{i,j} = c \frac{|x - x_{i*}|}{h_j} \Omega_{i,j},$$

since $|x - x_{j+1}| > |x - x_j|$. Thus, if we denote

$$\sigma_i(x) := \sum_{\nu : I_{\nu, n_1} \subseteq I_i} \tau_{\nu, n_1}(x),$$

then by (3.2) with $q = 0$, we have for $i \neq j$,

$$|\sigma_i(x)| \leqslant c \sum_{\nu : I_{\nu, n_1} \subseteq I_i} \frac{h_{\nu, n_1} \rho_1^r}{(\rho_1 + |x - x_{\nu, n_1}|)^{r+1}}$$
$$\leqslant c \frac{\rho_1^r}{(\rho_1 + |x - x_{i*}|)^{r+1}} \sum_{\nu : I_{\nu, n_1} \subseteq I_i} h_{\nu, n_1}$$
$$= \frac{c h_i \rho_1^r}{(\rho_1 + |x - x_{i*}|)^{r+1}}. \tag{3.10}$$

In the same way (with $q = 1$) we get for $i \neq j$,

$$|\sigma_i'(x)| \leqslant \frac{c h_i \rho_1^{r-1}}{(\rho_1 + |x - x_{i*}|)^{r+1}}. \tag{3.11}$$

Now by (3.1),

$$S'(x) - D_{n_1}'(x) = \sum_{i \neq j} [(p_j(x) - p_i(x))\, \sigma_i'(x) + (p_j'(x) - p_i'(x))\, \sigma_i(x)]$$
$$=: \sum_{i \neq j} \alpha_i(x),$$

and by virtue of (3.8) through (3.11),

$$|\alpha_i(x)| \leqslant \frac{c h_i}{h_j} \Omega_{i,j} \frac{\rho_1^{r-1}}{(\rho_1 + |x - x_{i*}|)^r}$$
$$\leqslant c a_{i,j} \frac{\varphi(\rho)}{\rho} \frac{h_i}{\rho_1} \left(\frac{\rho_1}{\rho_1 + |x - x_{i*}|} \right)^{r+2-3k}.$$

Hence

$$
\begin{aligned}
|S'(x)-D'_{n_1}(x)| &\leqslant \sum_{i=0,\, i\neq j}^{n} |\alpha_i(x)| \\
&= \sum_{i:\,|x-x_{i*}|<\delta} |\alpha_i(x)| + \sum_{i:\,|x-x_{i*}|\geqslant\delta} |\alpha_i(x)| \\
&\leqslant c a_k(S, \varphi; (x-\delta, x+\delta)\cap I)\,\frac{\varphi(\rho)}{\rho}\,\rho_1 \sum_{i=1}^{n} \frac{h_i}{(\rho_1+|x-x_{i*}|)^2} \\
&\quad + c a_k \frac{\varphi(\rho)}{\rho}\frac{1}{\rho_1} \sum_{i\neq j:\,|x-x_{i*}|\geqslant\delta} h_i \left(\frac{\rho_1}{\rho_1+|x-x_{i*}|}\right)^{r+2-3k} \\
&\leqslant c a_k(\delta)\,\frac{\varphi(\rho)}{\rho}\,\rho_1 \int_{-\infty}^{\infty} \frac{du}{(\rho_1+|x-u|)^2} \\
&\quad + c a_k \frac{\varphi(\rho)}{\rho}\,\rho_1^{r+1-3k}\, 2\int_{\delta}^{\infty} \frac{du}{(\rho_1+u)^{r+2-3k}} \\
&\leqslant c a_k(\delta)\,\frac{\varphi(\rho)}{\rho} + c a_k \frac{\varphi(\rho)}{\rho}\left(\frac{\rho_1}{\rho_1+\delta}\right)^{r+1-3k},
\end{aligned}
$$

where x being fixed, we used the shorter notation $a_k(\delta) := a_k(S, \varphi; (x-\delta, x+\delta))$. This concludes the proof of (3.6). ∎

The following lemma is crucial to our proof.

LEMMA 8. *Let the interval E consist of $l \geqslant 12s$ of the intervals I_j, and let $\mathscr{J}$ be a subcollection of $\mu \leqslant l/4$ of those intervals and we write $J := \bigcup \mathscr{J}$. Then for each $\varphi \in \Phi^k$, there exists a polynomial $Q_n(x) = Q_n(x; E; \mathscr{J}; \varphi)$, of degree not exceeding $30ksn$, satisfying*

$$Q'_n(x)\,\Pi(x) \geqslant 0, \qquad x \notin E\backslash(O\cup J); \tag{3.12}$$

$$Q'_n(x)\,\operatorname{sgn}\Pi(x) \geqslant -\frac{\varphi(\rho)}{\rho}, \qquad x \in E\backslash(O\cup J); \tag{3.13}$$

$$Q'_n(x)\,\operatorname{sgn}\Pi(x) \geqslant c_3\,\frac{l}{\mu}\frac{\varphi(\rho)}{\rho}, \qquad x \in J\backslash O, \tag{3.14}$$

where we may assume that $c_3 \leqslant 1$;

$$Q'_n(x)\,\operatorname{sgn}\Pi(x) \geqslant c_3\,\frac{l}{\mu}\frac{\varphi(\rho)}{\rho}\left(\frac{\rho}{\rho+\operatorname{dist}(x, E)}\right)^{36sk}, \qquad x \in I\backslash(E\cup O); \tag{3.15}$$

and

$$|Q_n(x)| \leqslant C\varphi(\rho)\, l^{k+1} 3^{kl} \left(\frac{|E|}{|E| + \operatorname{dist}(x, E)} \right)^3, \qquad x \in I, \tag{3.16}$$

where $|E|$ *denotes the length of* E.

Proof. We begin by quoting some results from [7] (see Lemma 5.3 there and the definitions above it). We put $b := \max(6s, 8k - s + 1)$, and for each j such that $I_j \cap O = \varnothing$ (which we denote by $j \in H$), we let the polynomials $T_j(x) = T_{j,n}(x; b; Y)$ and $\bar{T}_j(x) := \bar{T}_{j,n}(x; b; Y)$ of degree $(b+1)(4n-2) + s + 2$ be those defined there.

We recall that

$$T_j(1) = \bar{T}_j(1) = 1,$$

and that by [7, (5.15) and (5.16)] we have

$$T_j'(x)\, \Pi(x) \operatorname{sgn} \Pi(x_j) \geqslant 0, \qquad x \in I, \tag{3.17}$$

and

$$\bar{T}_j'(x)\, \Pi(x) \operatorname{sgn} \Pi(x_j) \leqslant 0, \qquad x \in I \backslash I_j. \tag{3.18}$$

Since

$$\left| \frac{x - y_i}{x_j - y_i} \right| \leqslant 1 + \left| \frac{x - x_j}{x_j - y_i} \right| < 4, \qquad x \in I_j,$$

[7, (5.19)] implies

$$|\bar{T}_j'(x)| \leqslant \frac{c}{\rho}, \qquad x \in I_j,$$

whence,

$$\varphi(h_j)\, |\bar{T}_j'(x)| \leqslant c\, \frac{\varphi(\rho)}{\rho}, \qquad x \in I_j. \tag{3.19}$$

If $\chi_j(x) := \chi_{[x_j, 1]}(x)$, then by [7, (5.23) and (5.24)], we have

$$|\chi_j(x) - T_j(x)| \leqslant c \left(\frac{h_j}{|x - x_j| + h_j} \right)^{8k+1}, \qquad x \in I,$$

and

$$|\chi_j(x)-\bar{T}_j(x)| \leqslant c\left(\frac{h_j}{|x-x_j|+h_j}\right)^{8k+1}, \qquad x\in I.$$

Hence, by virtue of (1.5), it follows that

$$\varphi(h_j)\,|\chi_j(x)-T_j(x)| \leqslant c\varphi(h_j)\,\frac{h_j}{\rho}\left(\frac{h_j}{|x-x_j|+h_j}\right)^4, \qquad x\in I,$$

and

$$\varphi(h_j)\,|\chi_j(x)-\bar{T}_j(x)| \leqslant c\varphi(h_j)\,\frac{h_j}{\rho}\left(\frac{h_j}{|x-x_j|+h_j}\right)^4, \qquad x\in I. \tag{3.20}$$

Similarly, by (1.6), we have (see also [7, (5.29) and (5.30)])

$$\varphi(h_j)\,|\chi_j(x)-T_j(x)| \leqslant \varphi(\rho)\,\frac{h_j}{\rho}\left(\frac{\rho}{|x-x_j|+\rho}\right)^4, \qquad x\in I, \tag{3.21}$$

and

$$\varphi(h_j)\,|\chi_j(x)-\bar{T}_j(x)| \leqslant \varphi(\rho)\,\frac{h_j}{\rho}\left(\frac{\rho}{|x-x_j|+\rho}\right)^4, \qquad x\in I. \tag{3.22}$$

In view of (1.2), it follows from [7, (5.28)] that

$$\varphi(h_j)\,|T_j'(x)| \geqslant c\,\frac{\varphi(\rho)}{\rho}, \qquad x\in I_j, \tag{3.23}$$

and observing that

$$\left|\frac{x-y_i}{x_j-y_i}\right| \geqslant \frac{|x-y_i|}{|x-x_j|+|x-y_i|} \geqslant \frac{\rho/3}{|x-x_j|+\rho}, \qquad x\in I\backslash O,$$

then finally (1.6) and [7, (5.22)] (see also [7, (5.27)]) yield

$$\varphi(h_j)\,|T_j'(x)| \geqslant c\,\frac{\varphi(\rho)}{\rho}\left(\frac{\rho}{|x-x_j|+\rho}\right)^{4b+s+k-2}, \qquad x\in I\backslash O. \tag{3.24}$$

We are ready to proceed with the proof. We write $j\in H(E)$ if $j\in H$ (defined at the beginning of the proof) and $I_j\subseteq E$. We denote by j^0 and j_0, the biggest and the smallest indices, respectively, in $H(E)$. Note that

$$\begin{aligned} |x-x_{j^0}|+\rho &\leqslant c(\mathrm{dist}(x,E)+\rho), \qquad x\leqslant x_{j^0},\\ |x-x_{j_0}|+\rho &\leqslant c(\mathrm{dist}(x,E)+\rho), \qquad x\geqslant x_{j_0}. \end{aligned} \tag{3.25}$$

Similarly we write $H(\mathscr{J}):=H(E)\cap\mathscr{J}$ and $H(\mathscr{J}^0):=H(\mathscr{J})\cup\{j^0, j_0\}$. Denote by l_1 the number of all indices in $H(E)$ and by μ_1 the number of indices in $H(\mathscr{J}^0)$. Then

$$l-3s\leqslant l_1\leqslant l, \qquad \text{and} \qquad \mu_1\leqslant\mu+2. \tag{3.26}$$

We set

$$\beta_j:=\operatorname{sgn}\Pi(x_j)$$

and consider two cases.

First assume that β_j has the same sign β, for all $j\in H(E)$ and let

$$a:=\frac{l}{\mu}\frac{\sum_{j\in H(\mathscr{J}^0)}\varphi(h_j)}{\sum_{j\in H(E)\setminus H(\mathscr{J})}\varphi(h_j)}. \tag{3.27}$$

We estimate the numerator by

$$\sum_{j\in H(\mathscr{J}^0)}\varphi(h_j)\leqslant(\mu+2)\max_{j\in H(E)}\varphi(h_j). \tag{3.28}$$

By (3.26), the number of elements in $H(E)\setminus H(\mathscr{J})$ is at least

$$l_1-\mu\geqslant l-3s-\mu\geqslant l/2.$$

Since $\varphi\in\Phi^k$, we can prove in a similar way to that of [9, Lemma 17.1] that

$$\sum_{j\in H(E)\setminus H(\mathscr{J})}\varphi(h_j)\geqslant cl\max_{j\in H(E)}\varphi(h_j).$$

Therefore we have established that a defined above is bounded, say by c_*, independently of E and $\mathscr{J}$. Hence by (3.19)

$$a\varphi(h_j)\,|\bar{T}_j'(x)|\leqslant c_*\frac{\varphi(\rho)}{\rho}, \qquad x\in I_j. \tag{3.29}$$

Put

$$Q_n(x):=\frac{\beta}{c_*}\left(\frac{l}{\mu}\sum_{j\in H(\mathscr{J}^0)}\varphi(h_j)\,T_j(x)-a\sum_{j\in H(E)\setminus H(\mathscr{J})}\varphi(h_j)\,\bar{T}_j(x)\right).$$

We will show that Q_n has the required properties. First, (3.12) follows immediately from (3.17) and (3.18). Also, by virtue of (3.17) and (3.18) we have

$$Q_n'(x) \operatorname{sgn} \Pi(x) \geqslant -\frac{a}{c_*} \varphi(h_j) \, |\bar{T}_j'(x)|, \qquad x \in I_j \subseteq E \setminus (O \cup J), \quad (3.30)$$

$$Q_n'(x) \operatorname{sgn} \Pi(x) \geqslant \frac{1}{c_*} \frac{l}{\mu} \varphi(h_j) \, |T_j'(x)|, \qquad x \in I_j \subset J \setminus O, \quad (3.31)$$

$$Q_n'(x) \operatorname{sgn} \Pi(x) \geqslant \frac{1}{c_*} \frac{l}{\mu} \varphi(h_{j_0}) \, |T_{j_0}'(x)|, \qquad x \geqslant x_{j_0 - 1}, \quad (3.32)$$

$$Q_n'(x) \operatorname{sgn} \Pi(x) \geqslant \frac{1}{c_*} \frac{l}{\mu} \varphi(h_{j^0}) \, |T_{j^0}'(x)|, \qquad x \leqslant x_{j^0}. \quad (3.33)$$

Now we obtain (3.13) from (3.29) and (3.30); (3.14) follows by virtue of (3.23) and (3.31); and (3.32), (3.33), (3.23), (3.24), and (3.25) are combined to yield (3.15).

Thus, to complete the proof we have to prove (3.16). To this end we rewrite Q_n as

$$\begin{aligned} c_* \beta Q_n(x) = & \Bigg(\frac{l}{\mu} \sum_{j \in H(\mathcal{J}^0)} \varphi(h_j)(T_j(x) - \chi_j(x)) \\ & - a \sum_{j \in H(E) \setminus H(\mathcal{J})} \varphi(h_j)(\bar{T}_j(x) - \chi_j(x)) \Bigg) \\ & + \Bigg(\frac{l}{\mu} \sum_{j \in H(\mathcal{J}^0)} \varphi(h_j) \, \chi_j(x) - a \sum_{j \in H(E) \setminus H(\mathcal{J})} \varphi(h_j) \, \chi_j(x) \Bigg) \\ =: & \, A(x) + B(x), \qquad \text{say.} \end{aligned}$$

If $x \in I \setminus E$, then all $\chi_j(x)$ with $j \in H(E)$ have the same value so that (3.27) implies that $B(x) = 0$. On the other hand, if $x \in E$, then by virtue of (3.27) and (3.28),

$$\begin{aligned} |B(x)| & \leqslant 2l(1 + 2/\mu) \max_{j \in H(E)} \varphi(h_j) \leqslant cl \max_{j \in H(E)} \varphi(h_j) \\ & \leqslant cl\varphi(\rho) \left(\frac{|E|}{\rho} \right)^{k/2} \leqslant cl^{k+1} 3^{kl} \varphi(\rho). \end{aligned} \quad (3.34)$$

In order to estimate $A(x)$ we first assume that $\rho \leqslant |E|$. Then we apply (3.21) and (3.22) and get

$$\begin{aligned} |A(x)| & \leqslant cl\rho^3 \varphi(\rho) \sum_{j \in H(E)} \frac{h_j}{(|x - x_j| + \rho)^4} \leqslant cl\rho^3 \varphi(\rho) \int_E \frac{du}{(|x - u| + \rho)^4} \\ & \leqslant cl\rho^3 \varphi(\rho) \frac{1}{(\operatorname{dist}(x, E) + \rho)^3} \leqslant cl\varphi(\rho) \left(\frac{|E|}{|E| + \operatorname{dist}(x, E)} \right)^3. \end{aligned} \quad (3.35)$$

If on the other hand, $|E|<\rho$, then by (3.20) we have

$$\begin{aligned}|A(x)| &\leqslant cl \sum_{j\in H(E)} \varphi(h_j)\frac{h_j^5}{\rho(|x-x_j|+h_j)^4}\\ &\leqslant cl\frac{\varphi(|E|)}{\rho}\sum_{j\in H(E)}\frac{|E|^4\,h_j}{(|x-x_j|+|E|)^4}\\ &\leqslant cl\,|E|^3\,\varphi(|E|)\frac{|E|}{\rho}\sum_{j\in H(E)}\frac{h_j}{(|x-x_j|+|E|)^4}\\ &\leqslant cl\,|E|^3\,\varphi(\rho)\int_E\frac{du}{(|x-u|+|E|)^4}\\ &\leqslant cl\,|E|^3\,\varphi(\rho)\frac{1}{(\operatorname{dist}(x,E)+|E|)^3}\\ &\leqslant cl\varphi(\rho)\left(\frac{|E|}{|E|+\operatorname{dist}(x,E)}\right)^3. \end{aligned} \tag{3.36}$$

This concludes the proof of (3.16).

In the second case, we assume that there are $j_1\in H(E)$ and $j_2\in H(E)$, such that $\beta_{j_1}\beta_{j_2}<0$. (In this case we do not make use of $\bar{T}_j$, and hence we can even strengthen (3.13) by replacing its right-hand side by zero.)

Depending on the sign of $\sum_{j\in H(\mathscr{J}^0)}\varphi(h_j)\,\beta_j$ we take $b\geqslant 0$ so that for

$$Q_n(x):=\frac{l}{\mu}\sum_{j\in H(\mathscr{J}^0)}\varphi(h_j)\,T_j(x)\,\beta_j+b\varphi(h_{j_i})\,T_{j_i}(x)\,\beta_{j_i},\qquad i=1 \text{ or } i=2,$$

we have $Q_n(1)=0$. Note that this implies that

$$b\leqslant\frac{l}{\mu}\frac{\sum_{j\in H(\mathscr{J}^0)}\varphi(h_j)}{\varphi(h_{j_i})}. \tag{3.37}$$

Now we proceed as in the first case and readily obtain

$$\begin{aligned} &Q_n'(x)\operatorname{sgn}\Pi(x)\geqslant 0, && x\in I,\\ &Q_n'(x)\operatorname{sgn}\Pi(x)\geqslant\frac{l}{\mu}\varphi(h_j)\,|T_j'(x)|, && x\in I_j\subset J\backslash O,\\ &Q_n'(x)\operatorname{sgn}\Pi(x)\geqslant\frac{l}{\mu}\varphi(h_{j_0})\,|T_{j_0}'(x)|, && x\geqslant x_{j_0-1},\\ &Q_n'(x)\operatorname{sgn}\Pi(x)\geqslant\frac{l}{\mu}\varphi(h_{j^0})\,|T_{j^0}'(x)|, && x\leqslant x_{j^0}. \end{aligned}$$

Finally, we rewrite

$$Q_n(x) = \left(\frac{l}{\mu} \sum_{j \in H(\mathscr{J}^0)} \varphi(h_j)(T_j(x) - \chi_j(x))\,\beta_j + b\varphi(h_{j_i})(T_{j_i}(x) - \chi_{j_i}(x))\,\beta_{j_i}\right)$$
$$+ \left(\frac{l}{\mu} \sum_{j \in H(\mathscr{J}^0)} \varphi(h_j)\,\chi_j(x)\,\beta_j + b\varphi(h_{j_i})\,\chi_{j_i}(x)\,\beta_{j_i}\right)$$
$$=: A(x) + B(x), \qquad \text{say}.$$

As before, if $x \in I \backslash E$, then $B(x) = 0$, and if $x \in E$, then by (3.37) (see (3.34))

$$|B(x)| \leqslant 2l\,\frac{\mu+2}{\mu} \max_{j \in H(E)} \varphi(h_j) \leqslant c l^{k+1} 3^{kl} \varphi(\rho).$$

The estimate of $|A(x)|$ is done in the same way as (3.35) and (3.36), where again we employ (3.37). ∎

4. PROOF OF THEOREMS 1 AND 2

The crux of the proof is a lemma the idea of which goes back to the seminal paper by DeVore [1].

LEMMA 9. *Let $\varphi \in \Phi^k$ and $S \in \Sigma_{k-1}$. Assume that*

$$a_k(S, \varphi) \leqslant 1, \tag{4.1}$$

and that

$$S'(x)\,\Pi(x) \geqslant 0, \qquad \textit{for} \quad x \in I \backslash O \quad \textit{or for} \quad x \in I \backslash O^*.$$

Then there is a polynomial P_n of degree $\leqslant cn$ such that

$$|P_n(x) - S(x)| \leqslant C\varphi(\rho_n(x)), \tag{4.2}$$

and

$$P_n'(x)\,\Pi(x) \geqslant 0, \qquad \textit{for} \quad x \in I \backslash O \quad \textit{or for} \quad x \in I \backslash O^*, \quad \textit{respectively}. \tag{4.3}$$

Proof. Set $r = 3k - 1 + 36sk$, and let $c_1 := C_1(k, r)$ of (3.6). We fix an integer c_4 so that

$$c_4 \geqslant \max(8k/c_3, 12s), \tag{4.4}$$

where c_3 is the constant of (3.14). Without loss of generality we are going to assume that n is divisible by c_4, i.e., $n =: Nc_4$, where this defines N.

We divide I into N intervals,

$$E_q := [x_{qc_4}, x_{(q-1)\,c_4}] = I_{qc_4} \cup \cdots \cup I_{(q-1)\,c_4+1}, \qquad q = 1, ..., N.$$

We will write $j \in UC$ (for "Under Control") if there is an $x \in I_j$ such that

$$|S'(x)| \leqslant 5c_1 \frac{\varphi(\rho)}{\rho}, \tag{4.5}$$

and we will say that $q \in G$ (for "Good"), if E_q contains at least $2k-3$ intervals I_j with $j \in UC$. Note that if $q \in G$, then (4.1) implies that any of the polynomials p_j, $(q-1)\,c_4+1 \leqslant j \leqslant qc_4$ satisfies

$$\left\| p'_j \frac{\rho}{\varphi(\rho)} \right\|_{E_q} \leqslant c. \tag{4.6}$$

Indeed, let $i \in UC$ and let $x^* \in I_i$ be such that (4.5) holds for x^*. Then by virtue of (4.1),

$$\begin{aligned} |p'_j(x^*)| &\leqslant |p'_j(x^*) - p'_i(x^*)| + |p'_i(x^*)| \\ &\leqslant \frac{c}{h_i} \|p_j - p_i\|_{I_i} + c\,\frac{\varphi(\rho_n(x^*))}{\rho_n(x^*)} \\ &\leqslant c\,\frac{\varphi(h_j)}{h_i} \left(\frac{h_{ij}}{h_j}\right)^k \leqslant c\,\frac{\varphi(h_j)}{h_j}, \end{aligned}$$

where we used the fact that when $(q-1)\,c_4+1 \leqslant i, j \leqslant qc_4$, then $h_i \sim h_j \sim h_{ij}$. Since there are at least $k-1$ intervals with $i \in UC$, which are not adjacent to each other, and p'_j is of degree $k-2$, we conclude that p'_j is bounded in E_q by the same bound. Again $\varphi(h_j)/h_j \sim \varphi(\rho)/\rho$ for any $x \in E_q$. This proves (4.6). In particular,

$$|S'(x)| \leqslant c\,\frac{\varphi(\rho)}{\rho}, \qquad x \in E_q. \tag{4.7}$$

We remark that once (4.7) holds in E_q, then it holds (with perhaps a bigger constant) in $E_{q+1} \cup E_q \cup E_{q-1}$.

Given any set $A \subseteq I$ denote

$$A^e := \bigcup_{I_j \cap A \neq \varnothing} I_j, \qquad A^{2e} := (A^e)^e, \qquad \text{and} \qquad A^{3e} := ((A^e)^e)^e. \tag{4.8}$$

Now set

$$E := \bigcup_{q \notin G} E_q, \tag{4.9}$$

and decompose S into a "small" part and a "big" one by setting

$$s_1(x) := \begin{cases} S'(x), & \text{if} \quad x \notin E^e \\ 0, & \text{if} \quad x \in E^e, \end{cases}$$

and $s_2 := S' - s_1$, and finally putting

$$S_1(x) := \int_{-1}^{x} s_1(u)\, du + S(-1), \qquad S_2(x) := \int_{-1}^{x} s_2(u)\, du.$$

We will show that

$$a_k(S_1, \varphi) \leqslant c, \tag{4.10}$$

which by virtue of (4.1) implies

$$a_k(S_2, \varphi) \leqslant c + 1 < [c+2] =: c_5. \tag{4.11}$$

To this end, put

$$p_{j1} := S_1|_{I_j},$$

then we will prove that

$$|S_1'(x) - p_{j1}'(x)| \leqslant c\, \frac{\varphi(h_j)}{h_j} \left(\frac{|x - x_j| + h_j}{h_j} \right)^{k-1}, \qquad x \in I. \tag{4.12}$$

We first observe that either $p_{j1}' \equiv 0$ or $p_{j1}' = p_j'$, and in the latter case (4.7) implies

$$|p_{j1}'(x)| \leqslant c\, \frac{\varphi(h_j)}{h_j}, \qquad x \in I_j.$$

Hence we always have

$$|p_{j1}'(x)| \leqslant c\, \frac{\varphi(h_j)}{h_j} \left(\frac{|x - x_j| + h_j}{h_j} \right)^{k-2}, \qquad x \in I. \tag{4.13}$$

Next we note that (4.7) is valid for S_1 and every $x \in I$, i.e.,

$$|S_1'(x)| \leqslant c\, \frac{\varphi(\rho)}{\rho}, \qquad x \in I. \tag{4.14}$$

Now, if $\rho \leqslant h_j$, then $\varphi(\rho) \leqslant \varphi(h_j)$ and (1.2) through (1.5) yield

$$\frac{\varphi(\rho)}{\rho} \leqslant \frac{\varphi(h_j)}{\rho} \leqslant c\,\frac{\varphi(h_j)}{h_j^2}\,(|x-x_j|+h_j),$$

and if $\rho \geqslant h_j$, then $\rho^{-k}\varphi(\rho) \leqslant h_j^{-k}\varphi(h_j)$, and (1.2) and (1.4) imply

$$\frac{\varphi(\rho)}{\rho} \leqslant \rho^{k-1}\,\frac{\varphi(h_j)}{h_j^k} \leqslant c\,\frac{\varphi(h_j)}{h_j}\left(\frac{|x-x_j|+h_j}{h_j}\right)^{(k-1)/2}.$$

Hence (4.14) yields

$$|S_1'(x)| \leqslant c\,\frac{\varphi(h_j)}{h_j}\left(\frac{|x-x_j|+h_j}{h_j}\right)^{k-1}, \qquad x \in I. \tag{4.15}$$

Combining (4.13) and (4.15) we have (4.12), and by it for $x \in I_i$,

$$\begin{aligned}|S_1(x)-p_{j1}(x)| &= \left|\int_{x_j}^{x} (S_j'(u)-p_{j1}'(u))\,du\right| \\ &\leqslant c\,\frac{\varphi(h_j)}{h_j}\left(\frac{h_{ij}}{h_j}\right)^{k-1}|x-x_j| \\ &\leqslant c\varphi(h_j)\left(\frac{h_{ij}}{h_j}\right)^{k},\end{aligned}$$

which is (4.10).

The set E is a union of disjoint intervals $F_p=[a_p, b_p]$, between any two of which there is an interval E_q with $q \in G$. We may assume that $n > c_4 c_5$ and we will write $p \in AG$ (for “Almost Good”) if F_p consists of no more than c_5 intervals E_q, i.e., if it consists of no more than $c_4 c_5$ intervals I_j. Set

$$F := \bigcup_{p \notin AG} F_p,$$

and let

$$s_4(x) := \begin{cases} S'(x), & \text{if} \quad x \in F^e \\ 0, & \text{otherwise,} \end{cases}$$

and $s_3 := S' - s_4$. (For the definition of F^e see (4.8).) Now put

$$S_3(x) := \int_{-1}^{x} s_3(u)\,du + S(-1), \qquad S_4(x) := \int_{-1}^{x} s_4(u)\,du.$$

Evidently S_3 and S_4 are comonotone with S in I, and proceeding as we did above we get

$$|S_3'(x)| \leqslant c \frac{\varphi(\rho)}{\rho}, \qquad x \in I, \tag{4.16}$$

and

$$a_k(S_4, \varphi) < c_6. \tag{4.17}$$

Now (4.16) together with Lemma 5 implies the existence of a polynomial V_n which is comonotone with S on $I \backslash O$ or on $I \backslash O^*$, as the case may be, such that

$$|S_3(x) - V_n(x)| \leqslant c\varphi(\rho), \qquad x \in I. \tag{4.18}$$

Since

$$s_4(x) = S'(x), \qquad x \in F^e,$$

then by (4.1) we have for $p \notin AG$,

$$a_k(S_4, \varphi; F_p^e) \leqslant a_k(S, \varphi; F_p^e) \leqslant a_k(S, \varphi) \leqslant 1. \tag{4.19}$$

Also for such p,

$$S_4'(x) = S_2'(x), \qquad x \in F_p^{3e}.$$

Hence from (4.11),

$$a_k(S_4, \varphi; F_p^{3e}) = a_k(S_2, \varphi; F_p^{3e}) \leqslant a_k(S_2, \varphi) < c_5. \tag{4.20}$$

We still have to approximate S_4. To this end we construct three polynomials Q_n and M_n of degree $< 30ksn$ and $D_{n_1}(S_4, \cdot)$ of degree n_1.

We begin with Q_n. For each q for which $E_q \subseteq F$, let $\mathscr{I}_q$ be the collection of intervals $I_j \subseteq E_q$ with $j \in UC$. Recall that $q \notin G$, therefore by (4.4), the number of such intervals is at most $2k - 4 < c_4/4$, and the total number of intervals in E_q is c_4. Thus Lemma 8 is applicable for each E_q and if we set

$$Q_n := \sum_{q\,:\,E_q \subseteq F} Q_n(\,\cdot\,; E_q; \mathscr{I}_q; \varphi),$$

where on the right-hand side are the polynomials guaranteed by Lemma 10, and denote

$$\mathscr{I} := \bigcup_{q\,:\,E_q \subseteq F} \mathscr{I}_q, \qquad J = \bigcup \mathscr{I},$$

then we conclude that Q_n satisfies

$$Q_n'(x)\,\Pi(x) \geqslant 0, \qquad x \in (I \backslash F) \cup O \cup J, \tag{4.21}$$

$$Q_n'(x)\,\operatorname{sgn}\Pi(x) \geqslant -\frac{\varphi(\rho)}{\rho}, \qquad x \in F \backslash (O \cup J), \tag{4.22}$$

and by (4.4),

$$Q_n'(x)\,\operatorname{sgn}\Pi(x) \geqslant 4\,\frac{\varphi(\rho)}{\rho}, \qquad x \in J \backslash O. \tag{4.23}$$

Note that (4.21), (4.22), and (4.23) follow since for any given x all relevant $Q_n'(x; E_q; \mathscr{J}_q; \varphi)$, except perhaps one, have the same sign. Finally it follows from (3.16) that

$$|Q_n(x)| \leqslant c\varphi(\rho), \qquad x \in I. \tag{4.24}$$

Next we define the polynomial M_n. For each F_p with $p \notin AG$, let $\mathscr{J}_{p-}$ denote the collection of three intervals in the left side of $F_p^{3e} \backslash \mathring{F}_p$, and let $\mathscr{J}_{p+}$ be the collection of three intervals in the right side of $F_p^{3e} \backslash \mathring{F}_p$. Similarly, let F_{p-} and F_{p+} be closed intervals each consisting of $l := c_4 c_5$ intervals I_j and such that $J_{p-} := \bigcup \mathscr{J}_{p-} \subset F_{p-} \subset F_p^{3e}$ and $J_{p+} := \bigcup \mathscr{J}_{p+} \subset F_{p+} \subset F_p^{3e}$. Now we set

$$M_n := \sum_{p \notin AG} \left(Q_n(\cdot\,; F_{p+}; \mathscr{J}_{p+}; \varphi) + Q_n(\cdot\,; F_{p-}; \mathscr{J}_{p-}; \varphi)\right).$$

Since $l = c_4 c_5$ and $\mu = 3$, it follows by (4.4) that $c_3 l/\mu \geqslant 4c_5$. Again we have by Lemma 8,

$$M_n'(x)\,\operatorname{sgn}\Pi(x) \geqslant -2\,\frac{\varphi(\rho)}{\rho}, \qquad x \in F \backslash O; \tag{4.25}$$

$$M_n'(x)\,\operatorname{sgn}\Pi(x) \geqslant 0, \qquad x \in O;$$

$$M_n'(x)\,\operatorname{sgn}\Pi(x) \geqslant 4c_5\,\frac{\varphi(\rho)}{\rho}, \qquad x \in F^{3e} \backslash (F \cup O): \tag{4.26}$$

and

$$M_n'(x)\,\operatorname{sgn}\Pi(x) \geqslant c_7\,\frac{\varphi(\rho)}{\rho}\left(\frac{\rho}{\operatorname{dist}(x, F^e)}\right)^{36sk}, \qquad x \notin F^{2e} \cup O. \tag{4.27}$$

Finally, it readily follows by virtue of (3.16) that

$$|M_n(x)| \leqslant c\varphi(\rho), \qquad x \in I. \tag{4.28}$$

The third auxiliary polynomial the properties of which we need to recall is $D_{n_1} := D_{n_1}(S_4, \cdot)$. By the choice of r and by (4.17), Lemma 7 yields

$$|S_4(x) - D_{n_1}(x)| \leqslant C_0 c_6 \varphi(\rho), \qquad x \in I, \tag{4.29}$$

and for any $\delta > 0$,

$$\begin{aligned} &|S_4'(x) - D_{n_1}'(x)| \\ &\quad \leqslant c_1 a_k(S_4; \varphi; (x-\delta, x+\delta)) \frac{\varphi(\rho)}{\rho} + c_8 \frac{\varphi(\rho)}{\rho} \left(\frac{\rho_{n_1}(x)}{\delta}\right)^{36ks}, \qquad x \in I, \end{aligned} \tag{4.30}$$

where $c_8 := C_2 c_6$. Noting that

$$\frac{\rho_{n_1}(x)}{\rho} \leqslant \frac{n}{n_1},$$

we are going to prescribe $n_1 = cn$ so big that

$$c_8 \left(\frac{n}{n_1}\right)^{36ks} \leqslant c_1 \min(1, 3c_5, c_7). \tag{4.31}$$

Now we write

$$R_n := D_{n_1} + c_1 Q_n + c_1 M_n,$$

and by virtue of (4.24), (4.28), and (4.29), we have

$$|S_4(x) - R_n(x)| \leqslant c\varphi(\rho), \qquad x \in I.$$

In view of (4.18), this proves (4.2) for $P_n := R_n + V_n$. Thus in order to conclude the proof of Lemma 9, we should prove that (4.3) holds for our P_n. To this end, we recall that V_n is comonotone with S where it is required so that we only have to deal with R_n. Since (4.30) holds with an arbitrary δ, we will prescribe different ones as needed. As long as $x \in F^{2e}$, it suffices to take $\delta := \rho$, while we recall (1.2) and the fact that both $x + \rho_n(x)$ and $x - \rho_n(x)$ are increasing in $I \backslash (I_1 \cup I_n)$. First assume that $x \in F$, so that $(x-\delta, x+\delta) \subseteq F^e$. If $x \in J \backslash O$, then $S_4'(x) \operatorname{sgn} \Pi(x) \geqslant 0$, and we obtain by (4.23), (4.25), (4.19), (4.30), and (4.31), that

$$
\begin{aligned}
R_n'(x) \operatorname{sgn} \Pi(x) &\geqslant c_1 Q_n'(x) \operatorname{sgn} \Pi(x) + S_4'(x) \operatorname{sgn} \Pi(x) \\
&\quad + c_1 M_n'(x) \operatorname{sgn} \Pi(x) - |S_4'(x) - D_{n_1}'(x)| \\
&\geqslant 4c_1 \frac{\varphi(\rho)}{\rho} - 2c_1 \frac{\varphi(\rho)}{\rho} - c_1 \frac{\varphi(\rho)}{\rho} - c_8 \frac{\varphi(\rho)}{\rho} \left(\frac{\rho_{n_1}(x)}{\rho}\right)^{36ks} \\
&\geqslant \frac{\varphi(\rho)}{\rho} (c_1 - c_8 (n/n_1)^{36ks}) \geqslant 0. \qquad (4.32)
\end{aligned}
$$

If, on the other hand, $x \in F \setminus (J \cup O)$, then (4.5) is violated and by virtue of (4.22), (4.25), (4.19), (4.30), and (4.31), we get

$$
\begin{aligned}
&R_n'(x) \operatorname{sgn} \Pi(x) \\
&\quad \geqslant S_4'(x) \operatorname{sgn} \Pi(x) + c_1 Q_n'(x) \operatorname{sgn} \Pi(x) \\
&\qquad + c_1 M_n'(x) \operatorname{sgn} \Pi(x) - |S_4'(x) - D_{n_1}'(x)| \\
&\quad \geqslant 5c_1 \frac{\varphi(\rho)}{\rho} - c_1 \frac{\varphi(\rho)}{\rho} - 2c_1 \frac{\varphi(\rho)}{\rho} - c_1 \frac{\varphi(\rho)}{\rho} - c_8 \frac{\varphi(\rho)}{\rho} \left(\frac{\rho_{n_1}(x)}{\rho}\right)^{36ks} \\
&\quad \geqslant \frac{\varphi(\rho)}{\rho} (c_1 - c_8 (n/n_1)^{36ks}) \geqslant 0. \qquad (4.33)
\end{aligned}
$$

Now assume that $x \in F^{2e} \setminus (F \cup O)$ so that $(x - \delta, x + \delta) \subseteq F^{3e}$. Again we have $S_4'(x) \operatorname{sgn} \Pi(x) \geqslant 0$, and by (4.21), (4.26), (4.20), (4.30), and (4.31), we obtain

$$
\begin{aligned}
R_n'(x) \operatorname{sgn} \Pi(x) &\geqslant c_1 Q_n'(x) \operatorname{sgn} \Pi(x) + S_4'(x) \operatorname{sgn} \Pi(x) \\
&\quad + c_1 M_n'(x) \operatorname{sgn} \Pi(x) - |S_4'(x) - D_{n_1}'(x)| \\
&\geqslant 4c_1 c_5 \frac{\varphi(\rho)}{\rho} - c_1 c_5 \frac{\varphi(\rho)}{\rho} - c_8 \frac{\varphi(\rho)}{\rho} \left(\frac{\rho_{n_1}(x)}{\rho}\right)^{36ks} \\
&\geqslant \frac{\varphi(\rho)}{\rho} (3c_1 c_5 - c_8 (n/n_1)^{36ks}) \geqslant 0.
\end{aligned}
$$

Finally, if $x \notin F^{2e} \cup O$, then we set $\delta := \operatorname{dist}(x, F^e)$, which implies that S_4' vanishes on $(x - \delta, x + \delta)$. Hence $a_k(S_4, \varphi; (x - \delta, x + \delta)) = 0$, so by (4.21), (4.27), (4.30), and (4.31), we conclude that

$$
\begin{aligned}
R_n'(x) \operatorname{sgn} \Pi(x) &\geqslant c_1 Q_n'(x) \operatorname{sgn} \Pi(x) + S_4'(x) \operatorname{sgn} \Pi(x) \\
&\quad + c_1 M_n'(x) \operatorname{sgn} \Pi(x) - |S_4'(x) - D_{n_1}'(x)| \\
&\geqslant c_1 c_7 \frac{\varphi(\rho)}{\rho} \left(\frac{\rho}{\delta}\right)^{36ks} - c_8 \frac{\varphi(\rho)}{\rho} \left(\frac{\rho_{n_1}(x)}{\delta}\right)^{36ks} \\
&\geqslant \frac{\varphi(\rho)}{\rho} \left(\frac{\rho}{\delta}\right)^{36ks} \left(c_1 c_7 - c_8 \left(\frac{\rho_{n_1}(x)}{\rho}\right)^{36ks}\right) \\
&\geqslant \frac{\varphi(\rho)}{\rho} \left(\frac{\rho}{\delta}\right)^{36ks} (c_1 c_7 - c_8 (n/n_1)^{36ks}) \geqslant 0. \qquad (4.35)
\end{aligned}
$$

Combining (4.32) through (4.35) we have constructed a polynomial satisfying (4.2) and (4.3). ∎

The proofs of Theorems 1 and 2 now follow from Lemmas 3, 4, and 9, except that in Lemma 9 the polynomial is of degree $\leqslant cn$. This is easily rectified. First we may assume that $c \geqslant k-1$ and then we replace n by $[n/c]$, and observe that

$$\rho_{[n/c]}(x) \leqslant 4c^2 \rho_n(x), \qquad x \in I,$$

thus

$$\varphi(\rho_{[n/c]}(x)) \leqslant \varphi(4c^2 \rho_n(x)) \leqslant 4^k c^{2k} \varphi(\rho_n(x)),$$

where we applied the fact that $\varphi \in \Phi^k$. Hence Theorems 1 and 2 hold for $n \geqslant c$, while for smaller n, see the remark after the statement of Theorem 2. ∎

5. A COUNTEREXAMPLE

In this section we prove Theorem 3 by providing an example (see a similar example in [7, Example 1.11]).

Let $y_1 := 1/(20n)$ and $y_2 := -1/(20n)$ and define

$$
f(x) := \begin{cases} 1, & x \leqslant y_2, \\ -20nx, & |x| < \dfrac{1}{20n}, \\ -1, & x \geqslant y_1. \end{cases}
$$

Then of course f is nondecreasing in $[-1, y_2]$ and $[y_1, 1]$; and nonincreasing in $[y_2, y_1]$. Note that $O^*(8\pi n, Y_n) \subset [y_1 - 1/4n, y_1 + 1/4n] \cup [y_2 - 1/4n, y_2 + 1/4n] \cup [-1, -1 + 1/n^2] \cup [1 - 1/n^2, 1] =: \tilde{O}(n, Y_n)$.

For $t < 1/(10n)$, we readily see that $\omega(f, t) = 20nt$. Let P_n be comonotone with f outside $\tilde{O}(n, Y_n)$ and contrary to (1.9) assume that

$$|f(x) - P_n(x)| \leqslant A\omega(f, \rho_n(x)), \qquad x \in [-1, 1].$$

Since $\rho_n(\pm 1 \mp 1/n^2) < 3/n^2 \leqslant 1/10n$, this implies

$$|f(\pm 1 \mp 1/n^2) - P_n(\pm 1 \mp 1/n^2)| < 60A/n \leqslant 1.$$

By virtue of the definition of f, we thus obtain $\pm P_n(\pm 1 \mp 1/n^2) < 0$. Now, P_n is nondecreasing in $[-1 + 1/n^2, -1/4n] \cup [1/4n, 1 - 1/n^2]$. Therefore we conclude that the norm of P_n in the interval $[-1 + 1/n^2, 1 - 1/n^2]$ is attained in $[-1/4n, 1/4n]$. Note that P_n is positive at $-1/4n$ and negative at $1/4n$, hence it vanishes somewhere inside, say at ζ. If $|P_n(\xi)| = \|P_n\| := \|P_n\|_{[-1+1/n^2, 1-1/n^2]}$, then $\zeta - \xi < 1/2n$, whence there exists θ such that

$$|P_n'(\theta)| > \left| \frac{P_n(\xi)}{1/2n} \right| = 2n \, \|P_n\|.$$

On the other hand, for $n \geqslant 2$, $2/(2 - 2/n^2) \leqslant 4/3$ and $\sqrt{1 - \theta^2} \geqslant \sqrt{1 - 1/16n^2} > 7/8$, thus by Bernstein's inequality for the interval $[-1 + 1/n^2, 1 - 1/n^2]$ we obtain

$$|P_n'(\theta)| \leqslant \frac{32n}{21} \, \|P_n\|,$$

which is a contradiction. This completes the proof of Theorem 3.

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